0.1. Definition: a derivative $f'(x)$ is the $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

0.2. Definition: if we have a function f and another function g s.t. the derivative of g is f, then we call g the anti-derivative of f.

1. Theorem (FTC): If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$ for all $x \in (a, b)$.

1.3. Lemma: Assume f is integrable over $[a, b]$. Define $G(y) = \int_a^y f(x)dx$ for all $y \in [a, b]$. If f is continuous at $y_0 \in (a, b)$, then G is differentiable at y_0 and $G'(y) = f(y)$.

Sketch of proof of lemma:
We write the derivative at some point of G with limit definition and substitute in terms of an integral of $f(x)$. Then we simplify a bit and invoke definition of continuity to see that $f(y) - \epsilon < f(x) < f(y) + \epsilon$. Then we plug these terms into an integral (and the inequality still holds) and a constant over an integral is simply the length of that integral times the constant, so we have $a < b < c$ where each term differs by $h \cdot \epsilon$. Then replugging into our limit definition we divide through by h, thus proving that we are within ϵ (precisely the definition), showing the limit exists and it is $f(y)$.

1.4. Proof of FTC:

By the Lemma, $G'(x) = f(x)$ and by assumption $g'(x) = f(x)$ which implies $(G(x) - g(x))' = 0$ which implies $G(x) - g(x) = C$ is a constant. Plug in $x = a$, $G(a) = \int_a^a f(x)dx = 0$ which implies $C = -g(a)$ and $G(x) - g(x) = -g(a)$ which implies $\int_a^b f(x)\,dx = G(b) = g(b) - g(a)$.

2. Corollary (CoV): If $: [a, b] \to [(a), (b)]$ is differentiable, and $'() > 0$ for all (inducing a bijection since it monotonically increases) and if f is continuous, then $\int_{(a)}^{(b)} f(x)dx = \int_a^b f(()) \cdot '()d$.

2.5. Example: $\int_0^1 \frac{1}{1+x} dx$

3. Corollary (IbP): If f and g are differentiable on $[a, b]$, then $\int_a^b f'(x)\, g(x)\, dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)\, g'(x)\, dx.$ Equivalently, $\int_a^b \big(f'(x)g(x) + f(x)g'(x)\big)\, dx = f(b)g(b) - f(a)g(a).$

3.6. Proof:

Since $\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x)$, FTC yields that $\int_a^b \big(f(x)g(x)\big)'\, dx = f(b)g(b) - f(a)g(a)$, which proves the result.